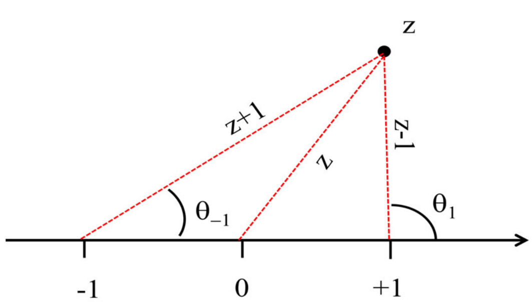
Mathematical Methods in Physics HW9

1. Consider the functions $w_1(z) = \sqrt{z}$ and $w_2(z) = \sqrt{(z^2-1)}$. Recall that a branch point is a point in the z-plane about which if we follow the values of $w(z)$ along a closed contour encircling it, then when we complete the contour we come back to a different value. Note, this should include circles small enough to avoid also encircling other branch points, as well as infinitesimal circles which pinpoint the identity of the branch point. For example if $z = 0$ is the only branch point, then a circle of radius 1 around $z = 0$ includes more than just $z = 0$, but only $z = 0$ is the branch point.

a) Determine the branch points for each function. One of the things that you should consider is whether $R = \infty$ constitutes a branch point. In order to figure this out, you can use an inversion map $z \to \frac{1}{\alpha}$ and then consider the newly formed function of α and whether $\alpha = 0$ is a branch point. *Hint: For w_1 and w_2 there are only 2.*

For $w_1(z)$ we know that it is singular at $z = 0$ and $z = \infty$ so these are the only two possible branch points. Let's first check $z = 0$ by going around a closed contour (circle) of radius 1 centered at the origin using the polar form $z = re^{i\theta}$: $w_1(\theta) = e^{\frac{i\theta}{2}} \Rightarrow w_1(0) = 1,\ w_1(2\pi) = -1$. So $z = 0$ is definitely a branch point. Now to check $z = \infty$ we replace $z \to \frac{1}{\alpha}$ and then express α in polar form and consider a circle of radius 1 around $\alpha = 0$, in which case:
$w_1\left(\frac{1}{\alpha}\right) = \frac{1}{\sqrt{r}} e^{-\frac{i\theta}{2}} \Rightarrow w_1(\theta) = e^{-\frac{i\theta}{2}} \Rightarrow w_1(0) = 1,\ w_1(2\pi) = -1$. Thus $z = \infty$ is also a branch point.

For $w_2(z) = \sqrt{(z^2-1)} = \sqrt{(z-1)(z+1)}$ we know that the three singular points are $z = \pm 1$ and $z = \infty$. To begin, it is easier for the points $z = \pm 1$ if we use a modified complex coordinate system. Imagine if we designed double polar coordinates around each of these singularities.

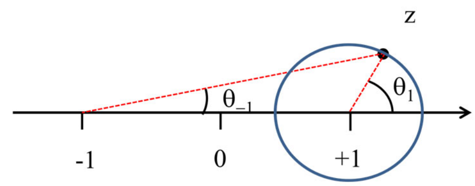


Now while it is clear that we can cover the entire z-plan with one radius and one angle coordinate, these allow us to simplify our expression. Note that the radius for each singularity is $|z \pm 1|$, and if we knew one of the angles and one of the radii, then we would be able to find out everything else, so these are not independent coordinates.

Using these we can now write the function as:

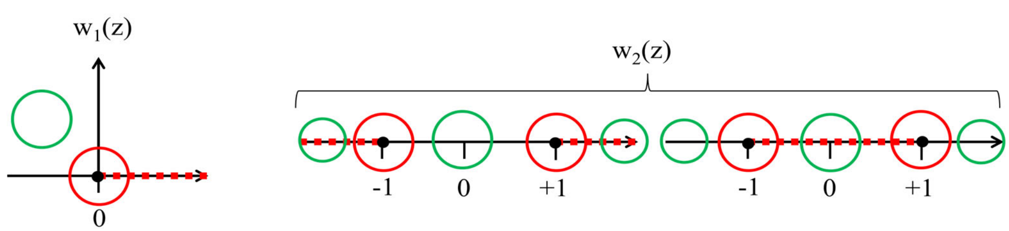
$$w_2(z) = \sqrt{|z-1|e^{i\theta_1}|z+1|e^{i\theta_{-1}}} = |z-1|^{\frac{1}{2}} e^{\frac{i\theta_1}{2}} |z+1|^{\frac{1}{2}} e^{\frac{i\theta_{-1}}{2}}$$

Now consider moving around a closed circle centered on $z = 1$ that does not include $z = -1$.



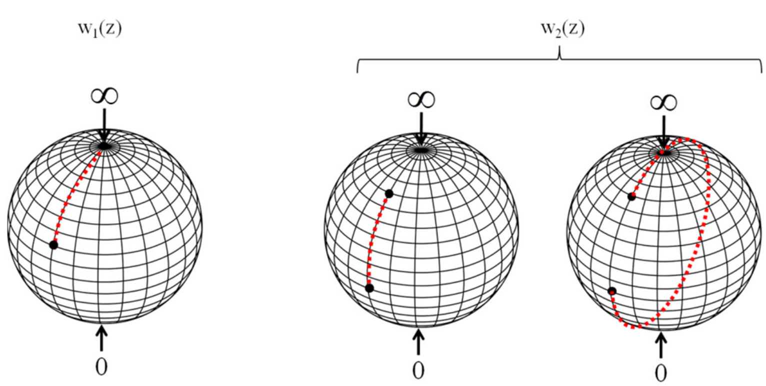
Then θ_1 goes from 0 to 2π, while θ_{-1} goes from 0 to $\pm\theta_{max}$ where $\theta_{max} < \frac{\pi}{2}$. Therefore according to the function, $w_2(z, \theta_1, \theta_{-1}) = -w_2(z, \theta_1 + 2\pi, \theta_{-1})$, which means that $z = 1$ is a branch point. A very similar argument holds for $z = -1$ by way of following a circle centered over it, which does not include $z = 1$, giving $w_2(z, \theta_1, \theta_{-1}) = -w_2(z, \theta_1, \theta_{-1} + 2\pi)$, and hence that $z = -1$ is a branch point as well. To determine whether $z = \infty$ is a branch point, one of the most simple things to do is imagine moving around a very large circle which does encircle ∞ as well as everything else. If the circle we are moving around includes both of the branch points at $z = \pm 1$ then both angles must go from 0 to 2π, and the negatives we get from each of them combine to give a positive. Thus $w_2(z, \theta_1, \theta_{-1}) = w_2(z, \theta_1 + 2\pi, \theta_{-1} + 2\pi)$ and hence the singularity at $z = \infty$ is not a branch point.

b) Now construct the branch cuts and label them on a drawing of the z-plane (along with the branch points). The rule in selecting where the cuts can go is that any curve that tries to go around a single branch point must pass through a branch cut. Can you connect the points with cuts and achieve this?



Notice that the red circles feel the effect of the branch cut while the green circles do not.

c) Finally, instead of thinking of the z-plane, consider instead the Riemann sphere which is just the surface of a two-sphere to which we can map the z-plane if we consider $R = 0$ at the south pole and if we include the point $R = \infty$ at the north pole. Draw the points and cuts using this representation of the complex plane. Does this help you understand the results for part (b)?



2. Compute the Laurent expansion of $w(z) = \frac{z^{122}+3z^{41}+1}{z^{568}}$ around $z = 0$. Use the formula, and then check your answer as we did in class.

$A_n = \frac{1}{2\pi i}\oint_C \frac{z^{122}+3z^{41}+1}{z^{568}z^{n+1}}dz = \frac{1}{2\pi i}\oint_C \frac{z^{122}+3z^{41}+1}{z^{n+569}}dz$

Consider $\oint_C \frac{1}{z^n}dz$. To evaluate use the polar form $z = re^{i\theta}$ with a closed path at fixed radius $|z| = R$.

Then $\int_0^{2\pi} \frac{1}{R^n e^{in\theta}} iRe^{i\theta}d\theta = \frac{i}{R^{n-1}}\int_0^{2\pi} e^{i(1-n)\theta}\, d\theta = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$

For $n = -568$ this becomes: $A_{-568} = \frac{1}{2\pi i}\oint_C \frac{z^{122}+3z^{41}+1}{z}dz = \frac{1}{2\pi i}\oint_C \left(z^{121} + 3z^{40} + \frac{1}{z}\right)dz = 0 + 0 + \frac{1}{2\pi i}\oint_C \frac{1}{z}dz = 1$

For $n = -527$ this becomes: $A_{-527} = \frac{1}{2\pi i}\oint_C \frac{z^{122}+3z^{41}+1}{z^{42}}dz = \frac{1}{2\pi i}\oint_C \left(z^{80} + \frac{3}{z} + \frac{1}{z^{42}}\right)dz = 0 + \frac{3}{2\pi i}\oint_C \frac{1}{z}dz + 0 = 3$

For $n = -447$ this becomes: $A_{-447} = \frac{1}{2\pi i}\oint_C \frac{z^{122}+3z^{41}+1}{z^{123}}dz = \frac{1}{2\pi i}\oint_C \left(\frac{1}{z} + \frac{3}{z^{82}} + \frac{1}{z^{123}}\right)dz = \frac{1}{2\pi i}\oint_C \frac{1}{z}dz + 0 + 0 = 1$

So we have:

$w(z) = \frac{1}{z^{568}} + \frac{3}{z^{527}} + \frac{1}{z^{447}}$

3. Compute the Laurent expansion of $w(z) = \frac{e^z}{z}$ around $z = 0$.

Starting with $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$, then $\frac{e^z}{z} = \sum_{n=0}^{\infty}\frac{z^{n-1}}{n!}$.

Alternatively we can use the Laurent expansion definition: $w(z) = \sum_{-\infty}^{\infty} A_n z^n$ with $A_n = \frac{1}{2\pi i}\oint_C \frac{w(z)}{z^{n+1}}dz$

Then $A_n = \frac{1}{2\pi i}\oint_C \frac{e^z}{z^{n+2}}dz$ and for $n < -1$ the integrand is analytic, and so the integral is zero. So we need only start with $n = -1$. Then our first few terms are:

$A_{-1} = \frac{1}{2\pi i}\oint_C \frac{e^z}{z}dz$, $A_0 = \frac{1}{2\pi i}\oint_C \frac{e^z}{z^2}dz$, $A_1 = \frac{1}{2\pi i}\oint_C \frac{e^z}{z^3}dz$, etc.

Now consider the Laurent expansion of $w'^{(z)} = e^z = \sum_{-\infty}^{\infty} A'_n z^n$ with $A'_n = \frac{1}{2\pi i}\oint_C \frac{e^z}{z^{n+1}}dz$.

But we also know that $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$, therefore $A'_n = \frac{1}{2\pi i}\oint_C \frac{e^z}{z^{n+1}}dz = \frac{1}{n!}$.

But $A'_n = A_{n-1}$ and hence $w(z) = \sum_{-\infty}^{\infty} A_n z^n = \sum_{-1}^{\infty}\frac{z^n}{(n+1)!}$ which is the same as above.

4. Evaluate the integral $I = \oint_C \frac{4z^3-1}{z(z-1)} dz$ around a contour centered around $z = 0$ and of radius $|z| = 5$.

Note that the contour encloses both singularities of the integrand, i.e. $z = 1$ and $z = 0$. Therefore we can use the residue approach, $I = \oint_C \frac{4z^3-1}{z(z-1)} dz = \oint_{C_1} \frac{4z^3-1/z}{z-1} dz + \oint_{C_0} \frac{4z^3-1/(z-1)}{z} dz$ where C_0 contains $z = 0$, but not $z = 1$, and C_1 contains $z = 1$, but not $z = 0$.

Then using the Cauchy integral formula $w(z_0) = \frac{1}{2\pi i} \oint_C \frac{w(z)}{z-z_0} dz$ to evaluate each of these:

$$I = \oint_C \frac{4z^3-1}{z(z-1)} dz = 2\pi i \left(\frac{4-1}{1} + \frac{-1}{-1}\right) = 8\pi i$$

5. Consider swapping the role of $z \leftrightarrow z^*$ in defining analytic functions. What needs to be changed in various definitions in order for the story to work out with this swap? *Hint: Recall that analytic functions are normally only functions of* z *and have the lovely property that closed contour integrals in regions where they are analytic always give zero.*

First of all we should redefine the primary derivative to be one with respect to z^*, i.e.

$$w'(z_0) = \lim_{z^* \to z_0} \frac{w(z^*) - w(z_0)}{z^* - z_0} = \lim_{\Delta z^* \to 0} \frac{w(z_0 + \Delta z^*) - w(z_0)}{\Delta z^*}.$$

Then for example the derivative of $w(z^*) = z^*$ works since $w'(z_0) = \lim_{\Delta z^* \to 0} \frac{z_0 + \Delta z^* - z_0}{\Delta z^*} = 1$ which has no dependence on approach.Then analytic functions at z_0 would be defined to have the this new derivative at and around a point z_0.

Now if we still call functions complex valued, then $w(z^*) = u(x,y) + iv(x,y)$ where $z^* = x - iy$. This leads to the modified Cauchy-Riemann relations:

$$w'(z_0) = \lim_{\Delta z^* \to 0} \left(\frac{\Delta u}{\Delta z^*} + i\frac{\Delta v}{\Delta z^*}\right) \begin{cases} \Delta y = 0, \Delta z^* = \Delta x \ \Rightarrow \ \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ \Delta x = 0, \Delta z^* = -i\Delta y \ \Rightarrow \ \frac{\partial u}{-i\partial y} + i\frac{\partial v}{-i\partial y} = i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

And finally, the integral along a contour C will be given by:

$$\int_C w(z^*) dz^* = \int_C [u(x,y) + iv(x,y)][dx - idy] = \int_C (udx + vdy) - i\int_C (udy - vdx)$$

Beyond these, everything else will hold.